
Why small portfolios are preferable and how to choose them



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Abstract

One of the fundamental principles in portfolio selection models is minimization of risk through diversification of the investment. However, this principle does not necessarily translate into a request for investing in all the assets of the investment universe. Indeed, following a line of research started by Evans and Archer almost 50 years ago, we provide here further evidence that small portfolios are sufficient to achieve almost optimal in-sample risk reduction with respect to variance and to some other popular risk measures, and very good out-of-sample performances. While leading to similar results, our approach is significantly different from the classical one pioneered by Evans and Archer. Indeed, we describe models for choosing the portfolio of a prescribed size with the smallest possible risk, as opposed to the random portfolio choice investigated in most of the previous works. We find that the smallest risk portfolios generally require no more than 15 assets. Furthermore, it is almost always possible to find portfolios that are just 1% more risky than the smallest risk portfolios and contain no more than 10 assets. The preference for small optimal portfolios is also justified by recent theoretical results on the estimation errors for the parameters required by portfolio selection models. Our empirical analysis is based on some new and on some publicly available benchmark data sets often used in the literature.

1. Introduction

Since the start of Modern Portfolio Theory in the 1950s, with the seminal mean-variance (MV) model of Markowitz,¹ the main aim of portfolio selection models was that of reducing the risk of an investment in the stock market through diversification while trying to achieve a satisfactory return.

However, an excessive diversification that requires investment in a large number of assets may be undesirable for several reasons. Indeed, starting from Markowitz and from the work of Evans and Archer,² it has been observed by several scholars that the decrease in in-sample volatility that one can obtain by increasing the number of securities in the portfolio rapidly declines with the size of the portfolio. In his fundamental book, Markowitz observed that: "To understand the general properties of large portfolios, we must consider the averaging together of large numbers of highly correlated outcomes. We find that diversification is much less powerful in this case. Only a limited reduction in variability can be achieved by increasing the number of securities in a portfolio."³ This intuition was confirmed in the very influential work where, for any given size K from 1 to 40, the authors randomly picked subsets of K assets from a market of 470 securities and computed some statistics on the standard deviations of the equally-weighted (EW) portfolios formed with each subset of assets.⁴ They found that the average standard deviation for each size K was decreasing and rapidly converging to an asymptote and they concluded that no more than around 10 assets were needed to almost completely eliminate the unsystematic variation of a portfolio return. Several authors have contributed to the debate about the right size of a portfolio that almost completely eliminates the diversifiable risk in a market.⁵ Furthermore, based on Evans and Archer's work and on other similar findings, such magic size, or size range, has been recommended in a number of textbooks on investment management and corporate finance.⁶

On a practical side, it is clear that acquiring and monitoring a portfolio that includes many assets is not suitable for small investors. Furthermore, even big investors should consider the opportunity cost of holding large portfolios and should

identify the threshold where the costs exceed the benefit of risk reduction.⁷ In addition, from a theoretical viewpoint, it has been shown that the estimation errors for the parameters needed to implement Markowitz's and other portfolio selection models may rapidly increase with the size of the portfolio when the historical data is limited.⁸ Such errors in the parameters are typically amplified in the optimization process required by the models, thus leading to unreliable solutions.⁹

All these observations, which will be better illustrated in the following, provide a motivation for selecting a relatively small portfolio among those optimizing an appropriate criterion. We thus provide models for selecting such small optimal portfolios and perform an empirical analysis to determine the best-size ranges for the portfolios depending on the size of the market. We also provide further evidence of the benefits of small portfolios, both in terms of in-sample risk reduction and in terms of out-of-sample performance. We point out that our approach is significantly different from the mainstream approach pioneered by Evans and Archer. Indeed, we overcome one of the main weaknesses of their approach, which consists of stating results that are valid only on average. In other words, if one picks an arbitrary EW portfolio of a given size in a market, there is no guarantee that its risk will not be much larger than the average risk of all portfolios of the same size in that market.

The conceptually simple solution that we propose here is just to choose the best EW portfolio for each given size with respect to variance, and, furthermore, the optimal portfolios for each given size with respect to three different and complementary risk measures. In this way, for each size we clearly obtain a portfolio which has a risk not greater (and typically, quite smaller) than the average risk. The reason why this simple idea was not investigated before is probably due to the computational hardness of the models required to find such best portfolios. Indeed, some of these models have been solved exactly only recently for small to medium-size markets,¹⁰ and one model is solved here for the first time. Once we have obtained the optimal size of the minimum risk portfolio, we proceed with a sensitivity analysis that allows us to find the smallest size of a portfolio whose risk is not more than 1% larger than that of the

1 Markowitz (1952, 1959)

2 Evans and Archer (1968)

3 Markowitz (1959, 102)

4 Evans and Archer (1968)

5 E.g., Newbould and Poon (1993) and references therein.

6 Tang (2004)

7 Statman (1987)

8 Kondor et al. (2007)

9 Michaud and Michaud (1998)

10 E.g., Angelelli et al. (2008); Cesarone et al. (2015) and references therein.

minimum risk portfolio, thus finding even smaller portfolios with satisfactory risk level.

Another difference between our approach and that of Evans and Archer consists of the possibility of using general weights in the selected portfolio instead of equal weights only. For each portfolio size, this clearly allows one to find portfolios with even lower in-sample risk. However, since optimizing weights may also cause the maximization of estimation errors,¹¹ this choice does not necessarily imply better out-of-sample performance. For both weighting schemes and for all risk measures, we find results comparable to those of Evans and Archer. More precisely, we identify some ranges of (typically small) sizes where the portfolio risks are minimized, and ranges of even smaller sizes where the portfolio risks do not exceed the minimum by more than 1%. The out-of-sample performance of the selected portfolios for each specified size is another important feature of our analysis which is rarely found in previous works on the subject. In our analysis, we find that the best performances are generally obtained by portfolios with no more than 15 assets.

As an interesting complement to our findings, we mention that, in a recent and detailed study on the empirical behavior of investors and on the performance of their portfolios, Ivković et al. show that portfolios of small investors with low diversification exhibit superior performance with respect to those with high diversification.¹²

2. Fixed-size portfolio models

In this section, we present a fixed-size version of several well-known portfolio selection models and we provide an integer or a mixed-integer linear or quadratic formulation for all models. We first need to introduce some notation. Let $T + 1$ be the length of the in-sample period used to estimate the inputs for the models. We use p_{it} to denote the price of the i th asset at time t , with $t = 0, \dots, T$; $r_{it} = \frac{p_{it} - p_{i,t-1}}{p_{i,t-1}}$ is the i th asset return at time t , with $t = 1, \dots, T$; x is the vector whose components x_i are the fractions of a given capital invested in asset i in the portfolio we are selecting; y is a Boolean vector whose components y_i are equal to 1 if asset i is selected, and 0 otherwise. We assume that n assets are available in a market and, adopting linear returns, we have that $r_{t,0,0} = \sum_{i=1}^n x_i r_{it}$ is the portfolio return at time t , with $t = 1, \dots, T$.

The n -dimensional vector μ is used to denote the expected returns of the n risky asset, while Σ denotes their covariance matrix, and u denotes an n -dimensional vector of ones.

2.1 The equally weighted portfolio

The most intuitive way to diversify a portfolio is to equally distribute the capital among all stocks available in the market. In terms of relative weights, we have $x_i = 1/n$. This is known as the EW (also called naïve or uniform) portfolio. Clearly, the choice of the EW portfolio does not use any in-sample information or involve any optimization approach. However, some authors claim that its practical out-of-sample performance is hard to beat on real-world data sets.¹³ Furthermore, from the theoretical viewpoint, Pflug et al. shows that when increasing the amount of portfolio model uncertainty, i.e., the degree of ambiguity on the distribution of the assets returns, the optimal portfolio converges to the EW portfolio.¹⁴ We will, thus use this portfolio as a benchmark to compare the performances of the portfolios obtained by the models.

2.2 Fixed-size minimum variance equally weighted portfolios

As already observed, the EW portfolio is the most robust choice when there is a great uncertainty about the distribution of the asset returns. However, the EW portfolio has the drawback of using all available assets, which may be too numerous and not all desirable. A first proposal to overcome this drawback comes from Jacob, who proposes to select a small EW portfolio (with a specified number K of assets) that has minimum variance among all EW portfolios of the same size.¹⁵ The model by Jacob is a nonlinear 0-1 optimization model that has not yet been tested in practice, owing to its computational complexity. Thanks to the recent advances in solution methods and computing power, here, we can propose an empirical study of such a fixed-size minimum variance equally weighted (FSMVEW) model, formally described below.

$$\begin{aligned} \min \quad & y^T \Sigma y \\ \text{s.t.} \quad & u^T y = K \\ & y \in \{0, 1\}^n \end{aligned} \tag{1}$$

This is probably the simplest fixed-size portfolio model and has the advantage of not requiring the problematic estimates of the assets' expected returns. Furthermore, the effects of the possible

11 DeMiguel et al. (2009b); Michaud (1989)

12 Ivković et al. (2008)

13 DeMiguel et al. (2009b)

14 Pflug et al. (2012)

15 Jacob (1974)

estimation errors of the covariance matrix Σ do not result in very large or small weights for some assets, but only influence the choice of the subset of selected assets in the portfolio. From the optimization viewpoint, it falls into the class of pseudo-Boolean quadratic programming problems, which are known to be theoretically hard to solve in the worst case (NP-hard).¹⁶ However, due to its special structure, practical problems of this type with several hundred variables can be solved fairly efficiently with available free or commercial codes. Note that the vector x of weights of the optimal FSMVEW portfolio selected by model (1) is obtained as $x = \frac{1}{K}y$. When $K = n$, the FSMVEW portfolio coincides with the EW portfolio.

2.3 Fixed-size minimum variance portfolios

Another model that does not require the estimates of the assets' expected returns is the extreme case of the Markowitz model, where we seek only to minimize variance. Within our framework, we thus consider the following fixed-size minimum variance (FSMV) model, where only K assets are allowed in the selected portfolio:

$$\begin{aligned} \min \quad & x^T \Sigma x \\ \text{s.t.} \quad & u^T x = 1 \\ & u^T y = K \\ & \ell y \leq x \leq y, \\ & y \in \{0, 1\}^n \end{aligned} \quad (2)$$

The first constraint above is the budget constraint; the second one represents the portfolio fixed-size constraint; u is an n -dimensional vector of ones; y is an n -dimensional vector of binary variables used to select the assets to be included in the portfolio; x is the vector of portfolio weights, and ℓ is a minimum threshold (often called the buy-in threshold) for the weights of the selected assets which must be greater than zero (in our experiments we chose $\ell = 0.01$). Without these thresholds, problem (2) could generate portfolios with less than K assets, which is equivalent to replacing the constraint $u^T y = K$ with $u^T y \leq K$. Note that problem (2) is a quadratic mixed integer programming (QMIP) problem that falls again in the class of NP-hard problems. However, also in this case, problems with a few hundred variables can be solved fairly efficiently with available free or commercial codes. Furthermore, a recently proposed specialized algorithm can solve problems of this type with up to 2,000 variables.¹⁷

¹⁶ Boros and Hammer (2002)
¹⁷ Cesarone et al. (2009, 2013)

2.4 Fixed-size minimum conditional value-at-risk portfolios

The fixed-size minimum conditional value-at-risk (FSMCMVaR) model is a minimum risk model like the previous one, but instead of variance it measures risk with conditional value-at-risk at a specified confidence level ε ($CVaR_\varepsilon$), namely the average of losses in the worst $100\varepsilon\%$ of the cases.¹⁸ In our analysis, losses are defined as negative outcomes, and we set ε equal to 0.05. The FSMCMVaR model can be written as follows:

$$\begin{aligned} \min \quad & CVaR_\varepsilon(x) \\ \text{s.t.} \quad & u^T x = 1 \\ & u^T y = K \\ & \ell y \leq x \leq y, \\ & y \in \{0, 1\}^n \end{aligned} \quad (3)$$

where ℓ plays the same role as in (2).

Using a classical approach introduced by Rockafellar and Uryasev,¹⁹ problem (3) can be reformulated as a mixed integer linear programming (MILP) problem with $n + T + 1$ continuous variables, n binary variables and $T + n + 3$ constraints. Some recent computational experiences reporting on the solution of this model with state-of-the-art commercial solvers show that models with more than a few hundred variables are hard to solve with general-purpose solvers and would probably benefit from more specialized methods.²⁰

2.5 Fixed-size minimum mean semi-absolute deviation portfolios

The last risk measure that we take into account in our analysis is the downside mean semi-absolute deviation (semi-MAD or SMAD):

$$SMAD(x) = E[\min(0, \sum_{i=1}^n (rit - \mu_i)x_i)], \quad (4)$$

This is a concise version of the more famous mean absolute deviation (MAD) risk measure, which is defined as the expected value of the absolute deviation of the portfolio return from its mean.²¹ Indeed, Speranza showed that semi-MAD leads to a portfolio selection model that is equivalent to the MAD model, but with half the number of constraints.²²

¹⁸ Acerbi and Tasche (2002)
¹⁹ Rockafellar and Uryasev (2000); Cesarone et al. (2014)
²⁰ Cesarone et al. (2015)
²¹ Konno and Yamazaki (1991)
²² Speranza (1993)

We thus consider the following fixed-size minimum semi-MAD (FSMSMAD) model:

$$\begin{aligned} \min \quad & SMAD(x) \\ \text{s.t.} \quad & u^T x = 1 \\ & u^T y = K \\ & \ell y \leq x \leq y, \\ & y \in \{0, 1\}^n \end{aligned} \tag{5}$$

where ℓ plays the same role as in (2). Using the linearization approach described,²³ we can reformulate this problem as a MILP problem with $n + T$ continuous variables, n binary variables and $n + T + 3$ constraints.²⁴ From the computational experiences reported,²⁵ it appears that this model, although slightly easier than the previous one, also cannot easily be solved with general purpose state-of-the-art solvers when more than a few hundred variables are involved.

3. Large estimation errors for large portfolios

In the real-life asset management, an important issue concerns the problem of the estimation errors of the inputs required by the portfolio selection models, and the subsequent amplification of these errors in the phase of risk-gain optimization.

To analyze the impact of the estimation and model errors on optimized portfolio weights, we recall here the methodology.²⁶ The authors study the sensitivity to estimation errors of several minimum risk portfolios by considering an artificial investment universe, represented by multivariate standard normal returns with different numbers n of assets and lengths T of the time series. They measure the effect of the estimation errors on portfolio selection with the following ratio:

$$q_0^2 = \frac{x_{opt}^T \hat{\Sigma} x_{opt}}{\hat{x}_{opt}^T \hat{\Sigma} \hat{x}_{opt}} \tag{6}$$

where Σ is the identity matrix, x_{opt} is the minimum variance portfolio w.r.t. an empirical covariance matrix Σ obtained from a simulation of the standard normal returns, and \hat{x}_{opt} is the minimum variance portfolio w.r.t. $\hat{\Sigma}$. Clearly, by construction, x_{opt} coincides with the EW portfolio both when short sellings are allowed and when they are restricted. Note that by optimality

²³ Speranza (1993)
²⁴ Cesarone et al. (2014)
²⁵ Cesarone et al. (2015)
²⁶ Kondor et al. (2007)

of x_{opt} w.r.t. Σ , the ratio q_0^2 will never be smaller than 1. An interesting asymptotic result for the ratio q_0 is provided by Kondor et al.,²⁷ who show that when $n, T \rightarrow +\infty$ with $n/T = \text{constant}$, the asymptotic average estimation error is:

$$\bar{q}_0 = \frac{1}{\sqrt{1 - \frac{n}{T}}} \tag{7}$$

To test the impact on the estimation error of the constraints that limit the number of selected assets (the fixed-size constraint) in the optimal portfolios, we replicate the experiment of Kondor et al. considering four artificial markets with $n = 100, 200, 400, 470$ and four lengths of the time series $T = 500, 1000, 1500, 2000$. We analyze the sensitivity to estimation errors of the following three portfolio models:

- ▶ Minimum variance with short selling (MV-ss)
- ▶ Minimum variance without short selling (MV-Noss)
- ▶ Fixed-size minimum variance (FSMV)

For the FSMV model (2) and for each artificial market with n assets, we consider 10 sizes of the selected portfolio $K = n \cdot (10\%, 20\%, \dots, 100\%)$.

In Figure 1, we report the mean of q_0 on 20 simulations for different values of n and T , as described above. The black solid line is the theoretical shape of q_0 as in (7), which is well fitted by the simulated black squares coming from the MV-ss model. The blue circles and the red crosses are the average values of q_0 obtained from the MV-Noss and FSMV models, respectively. It is clear that the addition of no short selling or size constraints to the portfolio selection models tends to reduce the influence of estimation errors, particularly when the portfolio becomes large, i.e., when n/T approaches the critical value 1, where the empirical estimate of the covariance matrix becomes singular. Furthermore, the size constraints seem to provide a slightly better reduction of the effects of the estimation errors on the models with respect to the no short selling constraints.

4. Empirical behavior of the models

In this section, we test the models described above on some publicly available data sets. The analysis consists of two parts. First, we examine the behavior of the portfolios selected by the

²⁷ Kondor et al. (2007)

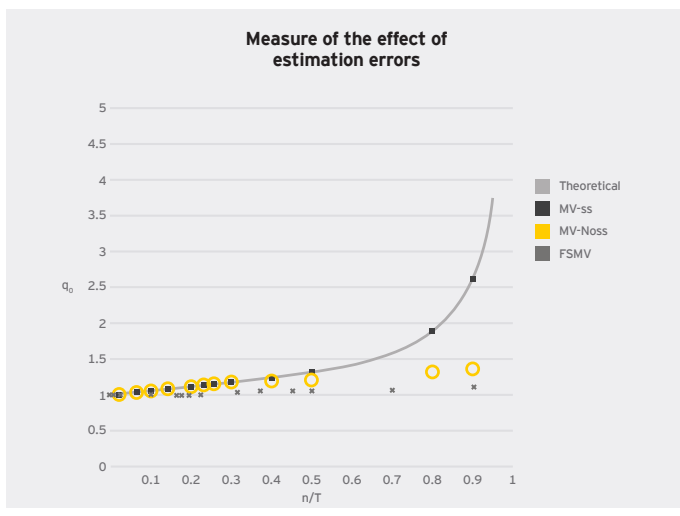


Figure 1: The average of q_0 as a function of n/T for different values of n and T .

models on the in-sample window, where we obtain the input parameters of the models. The second part consists of evaluating the out-of-sample performance of the portfolios, which is the aspect that matters most to investors. Since the markets are in continuous evolution, it seems appropriate to rebalance the portfolio from time to time in order to take new information into account. For this purpose, we use a rolling time window procedure (RTW), i.e., we shift the in-sample window (and consequently the out-of-sample window) all over the time length of each data set. More specifically, we consider a time window (in-sample period) of 200 observations for the data sets with weekly frequency, and of 120 observations for the data sets with monthly frequency. The choice of the lengths of the in-sample and the out-of-sample windows is based on typical settings of portfolio selection problems.²⁸ Then we solve the selection problem for overlapping windows built by moving forward in time with step size 4 (for the weekly data sets) or 1 (for the monthly data sets). The optimal portfolio found with respect to an in-sample period is held for the following four weeks (out-of-sample period of the weekly data sets) or one month (out-of-sample period of the monthly data sets). The out-of-sample performances of the resulting portfolios are evaluated in different ways by computing some performance measures commonly used in the literature.²⁹

28 E.g., Bruni et al. (2012, 2013); Cesarone et al. (2015); DeMiguel et al. (2009a)
29 Rachev et al. (2008)

Let $x^* = (x_1^*, \dots, x_n^*)$ denote the allocation of the selected portfolio and $r_t = (r_{1t}, \dots, r_{nt})$ denote the assets returns at time t . Then, in our analysis we consider:

- ▶ The **standard deviation** of the selected portfolio return
- ▶ The **Sharpe ratio** as $\frac{E[x^*r_t^* - r_f]}{\text{Std}[x^*r_t^* - r_f]}$, where $r_f = 0$
- ▶ The **Rachev ratio** as $\frac{CVaR_a[r_t - x^*r_t^*]}{CVaR_a[x^*r_t^* - r_f]}$, where $r_f = 0$ and $a = \beta = 0.1$
- ▶ The **max drawdown** as $-\min x^* r_t^*$, which is the maximum loss achieved by a portfolio during the holding period

In our analysis we use six data sets, summarized in Table 1. The monthly data sets (FF25, 48Ind, 100Ind) are taken from Ken French's website.³⁰

	Data set	# of assets	Time interval	Frequency	Source
1	FF25	25	07/1963-12/2004	Monthly	K. French
2	48Ind	48	07/1963-12/2004	Monthly	K. French
3	100Ind	100	01/1969-12/2011	Monthly	K. French
4	Stoxx50	32	01/2007-05/2013	Weekly	Yahoo Finance
5	FtseMib	34	01/2007-05/2013	Weekly	Yahoo Finance
6	Ftse100	63	01/2007-05/2013	Weekly	Yahoo Finance

Table 1: List of data sets analyzed.

4.1 In-sample analysis

For each model described in section 2, we study the behavior of its optimal value (minimum risk) when varying the number K of assets in the portfolio.

One of our main empirical findings is the scarce effect of diversification in terms of risk reduction when the portfolio size K does not belong to a certain range of values. Indeed, in all analyzed markets, we find that the risk measures, representing the objective functions of the models, achieve minimum values for a range of portfolio sizes corresponding to a significantly limited number of assets with respect to the total. Furthermore, these risk measures tend to increase when increasing the

30 The weekly data sets (Stoxx50, FtseMib, Ftse100) are downloaded from <http://finance.yahoo.com>, and are publicly available at <http://host.uniroma3.it/docenti/cesarone/DataSets.htm>.

portfolio size, thus contrasting the paradigm that the larger the diversification, the lower the risk. In Figures 2 and 3, we report some empirical evidence of this phenomenon for monthly (48Ind) and for weekly (Stoxx50) data sets. However, this behavior is similar for each data set analyzed. Figure 2 exhibits the boxplots of the different risk measures with respect to all considered in-sample windows by varying the portfolio size K . This means that, e.g., in the case of the 48Ind data set, for a fixed K we have 377 values of risk, one for each in-sample window (i.e., one for each rebalancing of the portfolio). Similarly, we obtain Figure 3, where we examine the Stoxx50 data set. Note, however, that in the cases of weekly data sets for a fixed K , we have 32 in-sample windows (i.e., 32 values of risk). As mentioned above, the boxplot of the in-sample volatility generated by the EW portfolios corresponds to that of the FSMVEW portfolios when $K = n$, and it generally presents the highest median volatility. This feature is common to all data sets, and it suggests that a greater diversification does not always imply a risk reduction, i.e., increasing the number of assets in the portfolio could worsen its in-sample performance in terms of risk.

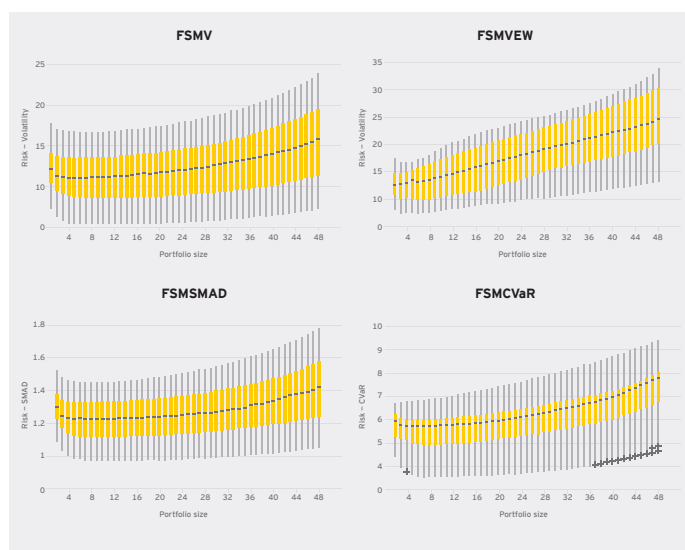


Figure 2: Boxplot of the in-sample risk w.r.t. the portfolio size for 48Ind.

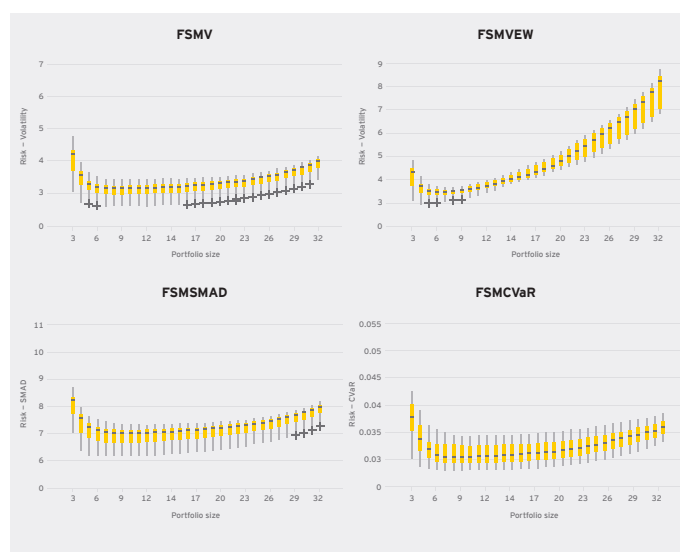


Figure 3: Boxplot of the in-sample risk w.r.t. the portfolio size for Stoxx50.

The empirical results of the FSMVEW portfolios could be compared to the findings obtained by Evans and Archer, and by further influential experiments in the literature, such as the well-known Fama's experiment.³¹ The author finds that, in a market with 50 stocks, the effect of *naïve diversification* determines a remarkable reduction of the portfolio in-sample volatility, but only when it is included in the portfolio of up to 20 stocks. We refer to *naïve diversification* as an EW strategy with a random selection of K out of n available stocks. Indeed, he observes that adding further stocks in the portfolio does not yield a considerable improvement. More precisely, Fama claims that approximately 95% of the possible reduction deriving from diversification is achieved passing from 1 to 20 assets. However, we point out that our approach is significantly different from that of Fama, as well as from that of Evans and Archer. Indeed, we overcome one of the main weaknesses of their approach, which consists of stating results that are valid only on average. In other words, if one picks an arbitrary EW portfolio of a given size in a market, there is no guarantee that its risk will not be much larger than the average risk of all portfolios of the same size in that market, whereas the results obtained by the FSMVEW portfolios are those corresponding to the best EW portfolios for each given size with respect to volatility. The findings on the FSMVEW model

31 Fama (1976)

also highlight that when the EW strategy is combined with risk minimization (instead of a random selection of K out of n available stocks), the selected small portfolios show an improvement in terms of both volatility and robustness of their values obtained on all in-sample windows.

In addition, once we have obtained the optimal size of the minimum risk portfolio for each in-sample window, we examine the range spanned by these optimal sizes. In Figure 4, we show for the 100Ind data set the distribution of the optimal portfolio sizes (i.e., corresponding to the global minimum risk) for all models analyzed with respect to all in-sample windows. We can see that the global minimum risk portfolio never exceeds 15 stocks for the 100Ind data set. However, this behavior is almost the same in all the other considered data sets, with the only exception of Ftse100, where the optimal portfolio size is seldom around 20 stocks. Furthermore, given the global minimum risk on an in-sample window, we detect the smallest size of a portfolio whose risk is not more than 1% larger than that of the minimum risk portfolio, thus finding even smaller portfolios with satisfactory risk level. Then, we repeat this procedure for each in-sample window and for each portfolio model. In Figure 5, we report the distribution of these 101% *min-risk* optimal portfolio sizes for each model analyzed with respect to all in-sample windows. More precisely, for each in-sample window we consider all the cardinalities for which the corresponding portfolio has a risk at most 1% greater than that of the minimum risk portfolio. As highlighted from the four sub-figures (one for each portfolio), the 101% *min-risk* portfolios generally show a significant risk reduction with 10 stocks for the 100Ind data set. Furthermore, in most of the cases we can achieve it with just six stocks. However, the 101% *min-risk* portfolio for the other data sets never exceeds a size of 15, and generally at most 10 stocks are needed.

The most compelling result emerging from the in-sample analysis is the existence of a portfolio size range (whose location could depend on the number of assets for each market), where one can generally find the lowest values of risk for all models considered. Indeed, we find that the smallest risk portfolios generally require no more than 15 assets. Furthermore, it is almost always possible to find portfolios that are just 1% more risky than the smallest risk portfolios and contain no more than 10 assets.

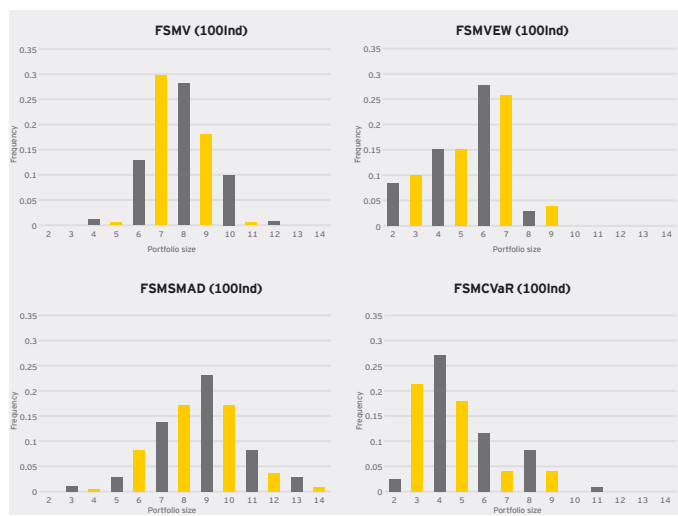


Figure 4: Distribution of the portfolio size corresponding to the global minimum risk.

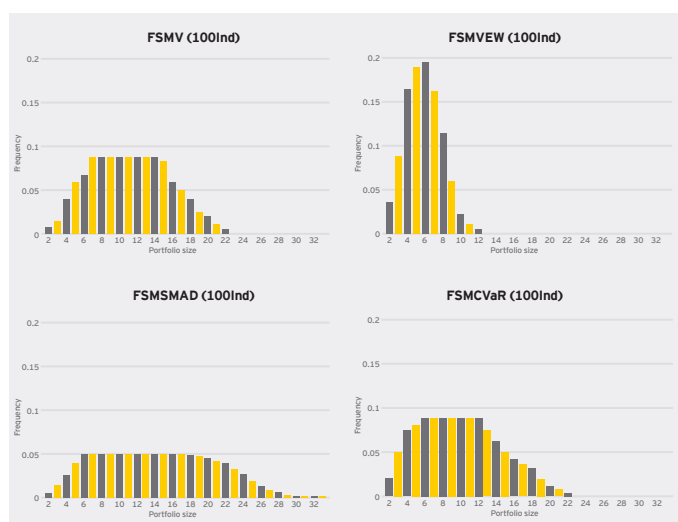


Figure 5: Distribution of the 101% *min-risk* portfolio.

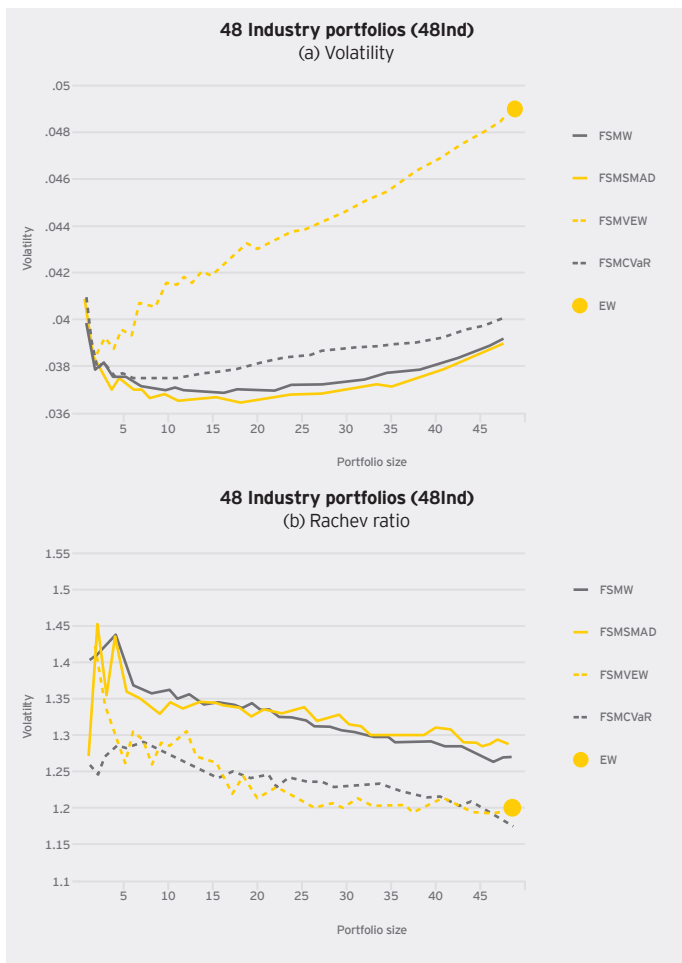


Figure 6: Analysis of the out-of-sample portfolio returns for 48Ind.

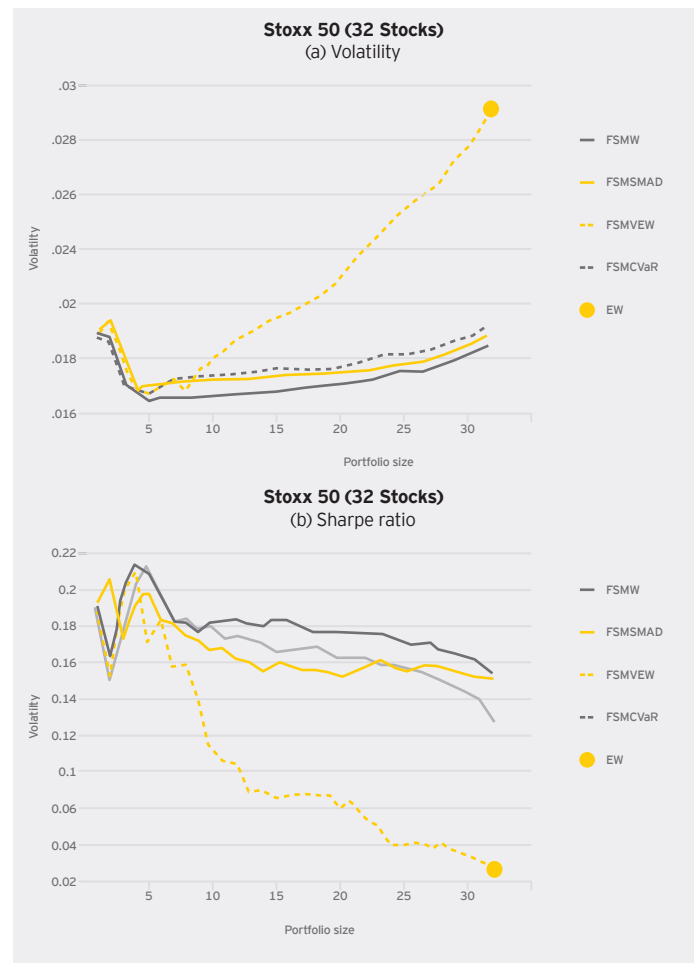


Figure 7: Analysis of the out-of-sample portfolio returns for Stoxx50.

4.2 Out-of-sample analysis

The second part of our analysis concerns the out-of-sample behavior of the portfolios. Our main goal is to confirm the finding, emerging from the in-sample analysis, that we can improve performances without investing in a large number of stocks.

Again, we consider the EW portfolio as a benchmark and, instead of focusing only on volatility reduction, we also compute the performance indices described in section 4. We start by verifying

the behavior of the out-of-sample standard deviation. More precisely, we check whether this performance measure reaches an optimal value, or at least a good value, for small-size portfolios. We can see in figures 6(a) and 7(a) that, both for monthly (48Ind) and for weekly (Stoxx50) data sets, the standard deviations of the portfolio returns reach their minima for small sizes. For larger sizes, the portfolio volatility tends to increase with different growth rates. These increases, except for the FSMVEW, are due to the buy-in threshold constraints. Without these constraints, we should expect nearly flat curves.

K = 5	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	0.0448	0.0378	0.0556	0.0134	0.0185	0.0169
FSMVEW	0.0431	0.0397	0.0424	0.0139	0.0181	0.0170
FSMV	0.0431	0.0377	0.0443	0.0141	0.0169	0.0166
FSMSMAD	0.0430	0.0377	0.0446	0.0136	0.0177	0.0172
EW	0.0509	0.0488	0.0512	0.0210	0.0296	0.0288
K = 10	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	0.0447	0.0376	0.0536	0.0133	0.0189	0.0175
FSMVEW	0.0441	0.0416	0.0428	0.0125	0.0203	0.0184
FSMV	0.0432	0.0372	0.0438	0.0126	0.0170	0.0168
FSMSMAD	0.0432	0.0369	0.0440	0.0128	0.0167	0.0174
EW	0.0509	0.0488	0.0512	0.0210	0.0296	0.0288
K = 15	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	0.0448	0.0378	0.0537	0.0134	0.0188	0.0179
FSMVEW	0.0458	0.0419	0.0435	0.0128	0.0218	0.0201
FSMV	0.0433	0.0371	0.0439	0.0127	0.0173	0.0170
FSMSMAD	0.0434	0.0367	0.0436	0.0126	0.0171	0.0175
EW	0.0509	0.0488	0.0512	0.0210	0.0296	0.0288

Table 2: Standard deviation of the out-of-sample returns.

K = 5	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	0.244	0.279	0.133	0.334	0.124	0.211
FSMVEW	0.267	0.285	0.249	0.247	0.119	0.165
FSMV	0.268	0.293	0.249	0.257	0.126	0.207
FSMSMAD	0.276	0.293	0.247	0.244	0.086	0.198
EW	0.276	0.242	0.215	0.173	-	0.026
K = 10	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	0.248	0.278	0.142	0.298	0.120	0.175
FSMVEW	0.290	0.268	0.256	0.284	0.091	0.099
FSMV	0.27	0.289	0.242	0.276	0.112	0.178
FSMSMAD	0.276	0.294	0.253	0.266	0.122	0.162
EW	0.264	0.242	0.215	0.173	-	0.026
K = 15	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	0.250	0.274	0.143	0.281	0.124	0.161
FSMVEW	0.296	0.260	0.249	0.278	0.042	0.072
FSMV	0.272	0.289	0.242	0.269	0.098	0.179
FSMSMAD	0.280	0.293	0.257	0.260	0.109	0.152
EW	0.264	0.242	0.215	0.173	-	0.026

Table 3: Sharpe ratio of the out-of-sample returns.

K = 5	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	1.216	1.281	0.877	1.618	1.162	1.167
FSMVEW	1.201	1.260	1.049	1.614	1.048	1.101
FSMV	1.245	1.403	1.043	1.514	1.164	1.205
FSMSMAD	1.25	1.363	1.032	1.382	1.07	1.218
EW	1.150	1.200	0.992	1.007	0.861	0.910
K = 10	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	1.215	1.271	0.892	1.459	1.145	0.861
FSMVEW	1.199	1.284	1.047	1.477	0.963	1.007
FSMV	1.244	1.360	1.027	1.514	1.147	1.150
FSMSMAD	1.234	1.343	1.033	1.333	1.147	1.185
EW	1.150	1.200	0.992	1.007	0.861	0.910
K = 15	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	1.217	1.245	0.896	1.397	1.123	1.064
FSMVEW	1.195	1.261	1.052	1.347	0.861	0.907
FSMV	1.244	1.343	1.024	1.426	1.135	1.135
FSMSMAD	1.234	1.343	1.047	1.320	1.103	1.132
EW	1.150	1.200	0.992	1.007	0.861	0.910

Table 4: Rachev ratio of the out-of-sample returns.

K = 5	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	0.209	0.150	0.578	0.027	0.042	0.031
FSMVEW	0.214	0.175	0.201	0.029	0.045	0.036
FSMV	0.198	0.123	0.206	0.030	0.037	0.031
FSMSMAD	0.201	0.121	0.206	0.039	0.041	0.032
EW	0.261	0.259	0.262	0.052	0.064	0.061
K = 10	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	0.210	0.145	0.501	0.033	0.042	0.034
FSMVEW	0.219	0.201	0.201	0.029	0.041	0.041
FSMV	0.198	0.126	0.209	0.023	0.039	0.031
FSMSMAD	0.203	0.125	0.212	0.031	0.039	0.034
EW	0.261	0.259	0.262	0.052	0.064	0.061
K = 15	FF25	48Ind	100Ind	Ftse100	FtseMib	Stoxx50
FSMCVaR	0.212	0.141	0.493	0.029	0.040	0.036
FSMVEW	0.235	0.206	0.213	0.027	0.050	0.045
FSMV	0.199	0.129	0.210	0.027	0.039	0.031
FSMSMAD	0.203	0.127	0.210	0.027	0.040	0.035
EW	0.261	0.259	0.262	0.052	0.064	0.061

Table 5: Max drawdown of the out-of-sample returns.

However, the buy-in threshold constraints are necessary to eliminate unrealistically small trades that can otherwise be included in an optimal portfolio. In Figures 6(b) and 7(b), we show the values of two other performance measures, namely the Rachev and Sharpe ratios, for the same data sets. As for the standard deviation, each model generally tends to provide the best values of the latter performance measures for small sizes. Furthermore, these values almost always decay when the portfolio size approaches n . This behavior provides further support for the idea of improving the performances of a portfolio by limiting the number of its stocks.

In addition to the graphical evidence, where only the most representative results are shown, we performed an extensive comparative analysis on all data sets considered. Since describing the results for all data sets and for all portfolio sizes is impractical, we report here the out-of-sample analysis for only three fixed sizes: $K = 5, 10, 15$. This choice is based on the observation that $K = 5, 10, 15$ generally belong to the optimal ranges in which the various models achieve the in-sample lowest risk for each data set. In Table 2, we provide the standard deviation of the out-of-sample returns for $K = 5, 10, 15$ for each model and data set analyzed. It is remarkable that the EW portfolio has almost always the worst performance, with the single exception of the 100Ind market, where the FSMCVaR portfolios generate the highest standard deviation. In Table 3, we report the Sharpe ratio of the out-of-sample returns for the same portfolio sizes of the previous table and for each model and data set analyzed. Note that when the portfolio excess return is negative, some gain-to-risk ratios have no meaning, thus we report "--". Again, we observe that the EW portfolio yields the worst performances compared with those of the other models, with the exception of the FSMCVaR portfolios for the 100Ind and FF25 markets. Similar considerations can be made about the Rachev ratio of the out-of-sample returns shown in Table 4. Indeed, again the EW portfolio tends to be the worst choice, with the only exception of the 100Ind data set. We also observe that for $K = 10$, the FSMV model seems to be preferable since it provides the best results for four data sets out of six, while for $K = 15$, it presents the best performances for five data sets out of six. The last performance measure considered in our analysis is the max drawdown, which is the worst out-of-sample loss achieved by a portfolio, as described in section 4. Table 5 shows that, again, the EW portfolio always has the worst performance for the prescribed sizes $K = 5, 10, 15$,

with the exception of the 100Ind market, where the FSMCVaR portfolios provide the worst loss. On the other hand, although no single model is clearly superior, we observe that the FSMV portfolios present the best values for three data sets out of six for $K = 5$, and for four data sets out of six for $K = 10$ and for $K = 15$.

5. Conclusions

The concept of diversification is not well defined and the measures of diversification are continuously evolving.³² However, the qualitative idea of diversification is to not overly concentrate the investments in very few stocks. Indeed, the role of diversification is to reduce risk by diversifying it as much as possible. In this work, we investigated the possible benefits and disadvantages of enlarging the portfolio size in several portfolio selection models with respect to various measures of performance. Similarly to various previous findings, but with a substantially different approach, our empirical results show that in most cases limiting the size of the selected portfolio improves both the in-sample and the out-of-sample performance. We might call this a small portfolio effect. These results are somewhat in line with the tendency described by DeMiguel et al.,³³ where an improved out-of-sample performance is often observed for the 1-norm-constrained minimum-variance portfolios. The analogy is based on the observation that the 1-norm is often regarded as an approximation of the 0-norm, i.e., the size of the portfolio. Further studies are underway to investigate the validity of this small portfolio effect with respect to other risk and performance measures and in larger markets.

32 E.g., Fragkiskos (2013); Meucci (2009) and references therein.

33 DeMiguel et al. (2009a)

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